2020
MATHEMATICS
[HONOURS]
Paper : III

Full Marks : 100 Time : 4 Hours

The figures in the right-hand margin indicate marks.
Notations and symbols have their usual meanings.

1. Answer any **three** from **(a)** to **(e)** and any **two** from the rest: $1 \times 5 = 5$

   a) Give an example to show that arbitrary union of compact sets may not be compact.

   b) If A is open and B is closed subset of $\mathbb{R}$ then show that A–B is open.

   c) Give an example of a function which is continuous everywhere but is not differentiable at two points.

   d) State monotone convergence theorem for sequence of real numbers.

   e) Let $f(x) = \sqrt{x},\ x \in [0, 2]$. Show that f is uniformly continuous but not Lipschitz continuous.

   f) Give an example of a commutative ring without multiplicative identity.

   g) Give an example of normal subgroup of S_3.

   h) Show that the system of three vectors (1, 3, 2), (1, –7, –8), (2, 1, –1) of $V_3(R)$ is linearly dependent.

2. Answer any **five** from **(a)** to **(f)** and any **five** from the rest: $2 \times 10 = 20$

   a) Show that the series $\sum_{n=1}^{\infty} \frac{n!}{n^n}$ is convergent.

   b) Prove that $[0, 1] = \bigcap_{n=1}^{\infty} \left(0, 1 + \frac{1}{n}\right)$.

   c) Prove that any non-empty open set of $\mathbb{R}$ is uncountable.

   d) Let $f : (0, 1) \to \mathbb{R}$ be defined by $f(x) = x \sin \frac{1}{x}$. Show that f is uniformly continuous.

   e) Show that $\{\cos n : n \in \mathbb{N}\}$ is dense in $[-1, 1]$.

   f) If $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$, then show that $\inf S = 0$.

g) If two abelian groups G and H are isomorphic then show that $\frac{G}{2G} \cong \frac{H}{2H}$.

h) Prove that if $d = \gcd(a, b)$, then $\frac{a}{d}$ and $\frac{b}{d}$ are integers prime to each other.

i) Let G be a group and let $a \in G$. Prove that $C(a) = C(a^{-1})$ where $C(a)$ denotes the centraliser of a.

j) Prove that a field does not contain any divisor of zero.

k) If n be a positive integer, prove that

$$\frac{1.3.7\ldots(2^n - 1)}{2.4.8\ldots 2^n} < \frac{2^n}{2^{n+1} - 1}.$$

l) Let $T: \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$T(x, y, z) = x(x + y + z).$$

Is T linear? Justify.

3. Answer any **three** from **(a)** to **(e)** and any **two** from the rest: 6×5=30

a) i) Show that between any two real numbers a and b $(a \neq b)$ there exists a rational number.

ii) Let S be a non-empty subset of $\mathbb{R}$ which is bounded below. Verify whether

$$\inf S = -\sup\{-s : s \in S\}.$$ 3+3

b) i) Show that the set of irrationals is neither open nor closed.

ii) Prove that if $f: \mathbb{R} \to \mathbb{R}$ has the intermediate value property and $f^{-1}(\{q\})$ is closed for every rational q, then f is continuous. 3+3

c) i) If $x_n = \sqrt{n}$, show that $(x_n)_{n=1}^{\infty}$ satisfies $\lim_{n\to\infty} |x_{n+1} - x_n| = 0$ but $(x_n)_{n=1}^{\infty}$ is not a Cauchy sequence.

ii) State Lagrange's MVT and give its geometric interpretation. 3+3

d) i) If $f, g: \mathbb{R} \to \mathbb{R}$ are continuous, then show that $\{x \in \mathbb{R} : f(x) \neq g(x)\}$ is an open set.

ii) Give an example of a divergent sequence which has exactly one cluster point. 3+3

e) If $a_n = \sin\frac{n\pi}{2} + \frac{(-1)^n}{n}$, $n \in \mathbb{N}$, then show that $\underline{\lim}_{n\to\infty} a_n = -1$ and $\overline{\lim}_{n\to\infty} a_n = 1$. 6

f) i) Prove that the number of primes is infinite.

ii) Prove that every finite integral domain is a field. 3+3

g) i) Find out the matrix of the linear transformation $T: \mathbb{R}^3 \to \mathbb{R}^3$ where

$$T(x, y, z) = (x + 2y, x + y + z, 2x + y)$$

with respect to the standard ordered basis.

ii) Show that $\mathbb{Z}_4 \not\cong K_4$ (K_4 denotes Klein 4-group). 3+3

h) Show that a non-zero finite ring having no divisor of zero is a ring with unity. 6

4. Answer any **three** questions: 15×3=45

a) i) If x, y, z are positive real numbers and x+y+z=1, prove that

$$8xyz \le (1-x)(1-y)(1-z) \le \frac{8}{27}.$$

ii) State and prove Fermat's Little Theorem.

iii) State and prove sequential criterion for continuity. 5+5+5

b) i) Show that $f: \mathbb{R} \to \mathbb{R}$ is continuous if and only if for every open set U of $\mathbb{R}$, $f^{-1}(U)$ is open.

ii) Define absolute convergence of a series. Prove that every absolutely convergent series is convergent. Give an example of a convergent series which is not absolutely convergent.

iii) Prove that every group is isomorphic to a group of permutations. 5+5+5

c) i) Let $S = \left\{ \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} : a, b \in \mathbb{R} \right\}$. Show that S is left but not right ideal of $M_2(\mathbb{R})$.

ii) Let $T: \mathbb{R}^3 \to \mathbb{R}^3$ be defined by

$$T(x, y, z) = (x - y,\ x + 2y,\ y + 3z),$$

$(x, y, z) \in \mathbb{R}^3$.

Show that T is invertible and determine T^{-1}.

iii) Verify that $\frac{x}{n(1+nx^2)}$ has a maximum at $x=\frac{1}{\sqrt{n}}$. Hence test for convergence of the series $\sum_{n=1}^{\infty}\frac{x}{(1+nx^2)}$ in $0<x<\infty$. 5+5+5

d) i) Show that the series $\sum_{n=1}^{\infty}\frac{1}{n^p}$ converges if $p>1$ and diverges if $p\le 1$.

ii) State Cauchy's General Principle for the existence of a finite limit of a real valued function $f(x)$ as $x\to a$, a being a finite real number. Use this principle to examine whether

$$\lim_{x\to 0}\left(\sin\frac{1}{x}+x\sin\frac{1}{x}\right)$$ exists or not.

iii) Examine if the ring of matrices

$$\left\{\begin{pmatrix} a & b \\ 2b & a \end{pmatrix} : a, b \in \mathbb{R}\right\}$$

contains divisor of zero. 5+6+4

e) i) Show that the following matrix is diagonalisable and find the diagonalised matrix:

$$\begin{pmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{pmatrix}$$

ii) If H is a normal subgroup of a group G then show that the binary operation on $\frac{G}{H}$ given by $aH.bH=abH$ is well-defined.

iii) If f and g are two continuous functions from $\mathbb{R}$ to $\mathbb{R}$ such that $f(x)<g(x)$ for all $x\in Q$ then show that $f(x)\le g(x)$ for all $x\in\mathbb{R}$. 5+5+5
